

Asosiy elementar funksiyalarning hosilalari

Yuqorida keltirilgan differensiallashning asosiy qoidalari yordamida hosila ta'rifi bo'yicha asosiy elementar funksiyalarning hosilaplarini keltirib chiqaramiz.

1. $y=c \Rightarrow c'=0$ ekanligini ko'rdik (c - o'zgarmas).

2. $y=a^x \Rightarrow (a \in R^+, a \neq 1)$.

$$\Delta y = a^{x+\Delta x} - a^x = a^x (a^{\Delta x} - 1),$$

$$\frac{\Delta y}{\Delta x} = a^x \cdot \frac{a^{\Delta x} - 1}{\Delta x}, \quad y' = a^x \lim_{\Delta x \rightarrow 0} \frac{a^{\Delta x} - 1}{\Delta x} = a^x \ln a.$$

Demak, $(a^x)' = a^x \ln a$.

Xususiyl holda, $a=e$ bo'lsa,

$$(e^x)' = e^x$$

ni olamiz.

3. $y=\log_a x \quad (a \in R^+, a \neq 1)$.

Bunga teskari funksiya $x=a^y$ ekanligidan yuqoridagi va teskari funksiya hosilasi formulasi asosida

$$y' = \frac{1}{x'} = \frac{1}{a^y \ln a} = \frac{1}{x \ln a} = \frac{\log_a e}{x}$$

ni olamiz. Demak,

$$(\log_a x)' = \frac{1}{x \ln a} = \frac{\log_a e}{x}$$

Xususiyl holda, $a=e$ bo'lsa,

$$(\ln x)' = \frac{1}{x}$$

kelib chiqadi.

4. $y=x^\alpha \quad (\alpha \neq 0)$. $x>0$ bo'lgan holda $x^\alpha = e^{\alpha \ln x}$ tenglik o'rinli ekanligi aniqdir.

Unga murakkab funksiyani differensiyalash qoidasini qo'llab,

$$(x^\alpha)' = (e^{\alpha \ln x})' = e^{\alpha \ln x} \cdot (\alpha \ln x)' = e^{\alpha \ln x} \cdot \alpha \cdot \frac{1}{x} = x^\alpha \cdot \frac{\alpha}{x} = \alpha x^{\alpha-1},$$

ya'ni

$$(x^\alpha)' = \alpha x^{\alpha-1}$$

ni olamiz.

Agar darajali funksiya $x<0$ bo'lganda ham aniqlangan bo'lsa, uning juftlik yoki toqlik xossasi asosida yuqoridagi formula o'rinli ekanligini keltirib chiqarish mumkin. Undan tashqari, $\alpha>1$ bo'lgan va darajali funksiya argumentning manfiy qiymatlari uchun ham aniqlangan holda $x=0$ nuqtada ham hosila mavjudligini va u yuqoridagi formula asosida aniqlanishini argumentning manfiy qiymati uchun aniqlanmagan bo'lsa, $x=0$ da o'ng hosila mavjud va u nolga tengligini aytamiz.

Agar $\alpha=1$ bo'lsa, $y=x$ funksiyaga ega bo'lamiz va bu holda $(x)'=1$ ekanligini olish qiyin emas. Demak, argument hosilasi birga teng ekan.

5. $y = \sin x$.

$$\Delta y = 2 \sin \frac{\Delta x}{2} \cdot \cos \left(x + \frac{\Delta x}{2} \right),$$

$$y' = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \left(\frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \cos \left(x + \frac{\Delta x}{2} \right) \right) = 1 \cdot \cos x,$$

ya'ni

$$(\sin x)' = \cos x$$

ni olamiz.

$$6. \quad y = \cos x. \quad \cos x = \sin \left(x + \frac{\pi}{2} \right).$$

$$(\cos x)' = \left(\sin \left(x + \frac{\pi}{2} \right) \right)' = \cos \left(x + \frac{\pi}{2} \right) \cdot \left(x + \frac{\pi}{2} \right)' = -\sin x \cdot (1 + 0) = -\sin x.$$

Demak,

$$(\cos x)' = -\sin x.$$

$$7. \quad y = \operatorname{tg} x, \quad x = \frac{\pi}{2} + \pi k, \quad k \in \mathbb{Z}; \quad \operatorname{tg} x = \frac{\sin x}{\cos x}.$$

Bo'linmani differensiallash qoidasini qo'llasak,

$$\begin{aligned} (\operatorname{tg} x)' &= \left(\frac{\sin x}{\cos x} \right)' = \frac{(\sin x)' \cdot \cos x - \sin x \cdot (\cos x)'}{\cos^2 x} = \frac{\cos x \cdot \cos x + \sin x \cdot \sin x}{\cos^2 x} = \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}, \end{aligned}$$

ya'ni

$$(\operatorname{tg} x)' = \frac{1}{\cos^2 x}$$

ni olamiz.

$$8. \quad y = \operatorname{ctg} x, \quad x = \pi k, \quad k \in \mathbb{Z}; \quad \operatorname{ctg} x = \frac{\cos x}{\sin x}.$$

Yuqoridagiga o'xshash,

$$(\operatorname{ctg} x)' = -\frac{1}{\sin^2 x}$$

ni olish mumkin.

$$9. \quad y = \arcsin x, \quad |x| \leq 1, \quad y \in \left[-\frac{\pi}{2}; \frac{\pi}{2} \right].$$

$$x = \sin y, \quad x' = \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

$$y' = \frac{1}{x'} = \frac{1}{\sqrt{1 - x^2}}, \quad |x| < 1$$

Demak,

$$(\arcsin x)' = \frac{1}{\sqrt{1 - x^2}}, \quad |x| < 1.$$

$$10. \quad y = \arccos x, \quad |x| \leq 1, \quad y \in [0; \pi].$$

Yuqoridagiga o'xshash,

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}, \quad |x| < 1$$

$$11. y = \operatorname{arctg} x. \quad x \in R, \quad y \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right).$$

$$x = \operatorname{tg} y, \quad x' = \frac{1}{\cos^2 y} = 1 + \operatorname{tg}^2 y = 1 + x^2,$$

$$y' = \frac{1}{x'} = \frac{1}{1+x^2}$$

Demak,

$$(\operatorname{arctg} x)' = \frac{1}{1+x^2}.$$

$$12. y = \operatorname{arcctg} x. \quad x \in R, \quad y \in (0; \pi).$$

Yuqoridagidek,

$$(\operatorname{arcctg} x)' = -\frac{1}{1+x^2}.$$

10.2.2. Hosila jadvali

Yuqorida olingan natijalarni quyidagicha joylashtiramiz.

$$1. (x^\alpha)' = \alpha x^{\alpha-1} \quad (\alpha \neq 0, \quad \alpha \in R).$$

$$(\sqrt{x})' = \frac{1}{2\sqrt{x}}; \quad \left(\frac{1}{x}\right)' = -\frac{1}{x^2}, \quad (x)' = 1.$$

$$2. (a^x)' = a^x \ln a \quad (a \in R^+, a \neq 1). \quad (e^x)' = e^x$$

$$3. (\log_a x)' = \frac{1}{x \ln a} = \frac{\log_a e}{x} \quad (a \in R^+, a \neq 1).$$

$$(\ln x)' = \frac{1}{x}.$$

$$4. (\sin x)' = \cos x.$$

$$5. (\cos x)' = -\sin x.$$

$$6. (\operatorname{tg} x)' = \frac{1}{\cos^2 x}.$$

$$7. (\operatorname{ctg} x)' = -\frac{1}{\sin^2 x}.$$

$$8. (\arcsin x)' = \frac{1}{\sqrt{1-x^2}}.$$

$$9. (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}.$$

$$10. (\operatorname{arctg} x)' = \frac{1}{1+x^2}.$$

$$11. (\operatorname{arcctg} x)' = -\frac{1}{1+x^2}.$$

Endi, yuqorida olingan differensiallash qoidalariga qo'shimcha sifatida *logarifmik differensiallash usulini* keltiramiz. Bu usul $y=f(x)$ berilgan bo'lsa, uni avvalo logarifmlab, so'ngra

$$(\ln y)' = \frac{1}{y} \cdot y'$$

tenglikdan

$$y' = y(\ln y)'$$

olinadigan formulaga asoslanadi.

Bunga misol tariqasida $y=U^g$ ning hosilasini topishni ko'ramiz. Bu yerda U va g lar differensiallanuvchi funksiyalar bo'lib, $U>0$ deb faraz qilamiz. U holda,

$$\ln y = g \ln U$$

Bundan

$$(\ln y)' = g' \ln U + g \cdot \frac{1}{U} \cdot U',$$

yuqoridagi formula asosida

$$(U^g)' = U^g \left(\frac{g}{U} \cdot U' + g' \ln U \right)$$

yoki

$$(U^g)' = g U^{g-1} \cdot U' + (\ln U) \cdot U^g \cdot g'$$

Eslatma. $y=U^g$ funksiya, U va g lar ikkalasi ham o'zgaruvchi funksiyalar bo'lgan holda *darajali-ko'rsatkichli* (yoki *ko'rsatkichli-darajali*) funksiya deb yuritiladi. Agar U -o'zgaruvchi g - o'zgarmas funksiyalar bo'lsa, uni murakkab darajali, U -o'zgarmas g o'zgaruvchi bo'lgan holda murakkab ko'rsatkichli funksiya deb yuritiladi. Yuqoridagi so'ngi formulada o'ng tomondagi ikkala qo'shiluvchilarga e'tibor bersak, birinchisi U^g ni murakkab darajali, ikkinchisi esa murakkab ko'rsatkichli deb faraz qilib olingan hosilalar ekanligini payqash qiyin emas.

Logarifmik differensiallash usuli funksiyalarning ko'paymasi, bo'linmasi, darajaga ko'tarish va ildiz chiqarish amallari qatnashgan funksiyalar hosilalarini topishda qo'l keladi.

Masalan, $y = \prod_{i=1}^n U_i$ bo'lib, U_i ($i = \overline{1;n}$) lar musbat va differensiallanuvchi funksiyalar bo'lsa,

$$\ln y = \sum_{i=1}^n \ln U_i$$

$$(\ln y)' = \sum_{i=1}^n \frac{U_i'}{U_i},$$

$$y' = y(\ln y)' = \prod_{i=1}^n U_i \cdot \sum_{j=1}^n \frac{U_j'}{U_j} = U_1' U_2 \dots U_n + U_1 U_2' \dots U_n + \dots + U_1 U_2 \dots U_n'$$

ya'ni

$$\left(\prod_{i=1}^n U_i \right)' = U_1' U_2 \dots U_n + U_1 U_2' \dots U_n + \dots + U_1 U_2 \dots U_n'$$

formulani olamiz.

Xususiyl holda, $n=3$ desak,

$$(U_1 U_2 U_3)' = U_1' U_2 U_3 + U_1 U_2' U_3 + U_1 U_2 U_3'.$$

10.3. Hosilaning fizik va geometrik ma'nolari

Avvalo, hosilaning fizik ma'nosini qaraylik. Aytaylik, moddiy nuqta to'g'ri chiziqli harakatda bo'lib uning t paytdagi bosib o'tgan yo'li $S=S(t)$ funksiya qiymati orqali ifodalansin. Agar t ga Δt ortirma berib, shu Δt vaqt oralig'ida moddiy nuqta bosib o'tgan yo'lni qarasak, u

$$\Delta S = S(t + \Delta t) - S(t)$$

funksiya orttirmasidan iborat ekanligi ayondir.

Endi, shu Δt vaqt oralig'idagi moddiy nuqtaning o'rtacha tezligini qarasak, u

$$\bar{g} = \frac{\Delta S}{\Delta t}$$

ekanligi ham o'quvchiga fizikadan ma'lum. Undan tashqari, moddiy nuqtaning t paytdagi oniy (haqiqiy) tezligi

$$g = \lim_{\Delta t \rightarrow 0} \bar{g}$$

ekanligi ham ma'lum. Demak,

$$g = \lim_{\Delta t \rightarrow 0} \bar{g} = \lim_{\Delta t \rightarrow 0} \frac{\Delta S}{\Delta t} = S'(t),$$

ya'ni, moddiy nuqta to'g'ri chiziqli harakatda bo'lsa, bosib o'tayotgan yo'lning funksiyasidan olingan hosila uning qaralayotgan paytdagi tezligidan iborat ekanligi kelib chiqadi. Bu hosilaning fizik ma'nosidir.

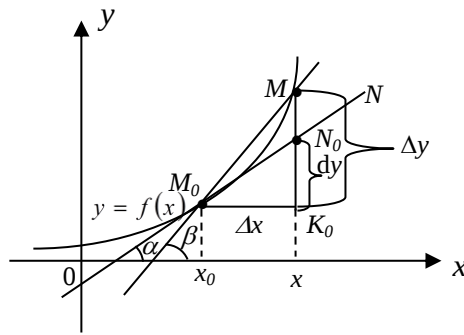
Endi, hosilaning geometrik ma'nosini keltirib chiqaramiz. Buning uchun, agar $y=f(x)$ funksiya x_0 nuqtada differensiallanuvchi bo'lsa, bu nuqtaning qandaydir atrofida aniqlangan ekanligidan shu atrofda uning grafigini quramiz.

Aytaylik, x_0 nuqtaga grafikning $M_0(x_0; y_0)$ nuqtasi $x_0 + \Delta x$ ga esa $M(x_0 + \Delta x, y_0 + \Delta y)$ nuqtasi mos kelsin. U holda, 10.3.1-rasmdan

$$\operatorname{tg} \beta = \frac{K_0 M}{M_0 K_0} = \frac{\Delta y}{\Delta x}$$

bo'lib, $\Delta x \rightarrow 0 \Rightarrow M \rightarrow M_0 \Rightarrow \beta \rightarrow \alpha$ va $y' = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\beta \rightarrow \alpha} \operatorname{tg} \beta = \operatorname{tg} \alpha$, bu yerda β - $M_0 M$

kesuvchining, α esa $M_0 N$ urinmaning OX o'qi yo'nalishi bilan hosil qilgan burchagidir. Agar $M \rightarrow M_0 \Rightarrow MM_0 \rightarrow MN$, deb faraz qilsak, MN to'g'ri chiziq M_0 nuqtadagi urinma deb atalib, kesuvchi M_0 nuqtadagi urinma holatiga yaqinlashadi.



10.3.1-расм.

Bundan ko‘rinadiki, funksiya hosilasi qaralayotgan nuqtadagi grafikka o‘tkazilgan urinmaning burchak koeffitsientidan iborat bo‘lar ekan. Bu hosilaning geometrik ma‘nosidir.

Bundan foydalanib, funksiya grafigiga o‘tkazilgan urinma tenglamasini yozish mumkin:

$$y - f(x_0) = f'(x_0) (x - x_0).$$

Xuddi shunga o‘xshash, qaralayotgan nuqtadagi normal (urinmaga perpendikulyar to‘g‘ri chiziq) tenglamasi

$$y - f(x_0) = -\frac{1}{f'(x_0)} (x - x_0)$$

bo‘ladi (bu yerda $f'(x_0) \neq 0$ deb faraz qilindi).